GIRLS' HIGH SCHOOL AND COLLEGE, PRAYAGRAJ

2020 – 2021

CLASS - 12 B & C

MATHEMATICS

WORKSHEET NO. 6

CHAPTER: INVERSE TRIGNOMETRIC FUNCTIONS

CONTINUITY AND DIFFERENTIABLITY

Note: Parents are expected to ensure that the student spends two days to read and understand the topic according to the book or the website referred and thereafter answer the questions

Book: ISc. Mathematics for class 12 by OP Malhotra

Website: www.khanacademy.org, www.topperlearning.com or any other relevant website.

Property 1

i. $\sin^{-1}(\sin\theta) = \theta$ for all $\theta \in [-\pi/2, \pi/2]$
ii. $\cos^{-1}(\cos\theta) = \theta$ for all $\theta \in [0, \pi]$
iii. $\tan^{-1}(\tan\theta) = \theta$ for all $\theta \in (-\pi/2, \pi/2)$

Property 2

i. $\sin^{-1}(1/x) = \operatorname{cosec}^{-1}x$, $x \geq 1$ or $x \leq -1$

ii. $\cos^{-1}(1/x) = \sec^{-1}x$, $x \geq 1$ or $x \leq -1$

iii. $\tan^{-1}(1/x) = \cot^{-1}x$, $x > 0$

Proof : $\sin^{-1}(1/x) = \operatorname{cosec}^{-1}x$, $x \geq 1$ or $x \leq -1$,
Let $\sin^{-1}\frac{1}{x} = y$ -------------------------(i)
$\frac{1}{x} = \sin y$

i.e. $x = \operatorname{cosec} y$

$\csc^{-1} x = y$

$\csc^{-1} x = \sin^{-1}\frac{1}{x}$ from (i)

where, $x \geq 1$ or $x \leq -1$.

Property 3

i. $\sin^{-1}(-x) = -\sin^{-1}(x)$, $x \in [-1,1]$

ii. $\tan^{-1}(-x) = -\tan^{-1}(x)$, $x \in R$

iii. $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1}(x)$, $|x| \geq 1$

Proof: $\sin^{-1}(-x) = -\sin^{-1}(x)$, $x \in [-1,1]$
Let, $\sin^{-1}(-x) = y$
Then $-x = \sin y$
$x = -\sin y$

$x = \sin(-y)$
$\sin^{-1} x = -y$

$\sin^{-1} x = -\sin^{-1}(-x)$

$\Rightarrow \sin^{-1}(-x) = -\sin^{-1}(x)$

$\sin^{-1} x \in [-1,1]$

Property 4

i. $\cos^{-1}(-x) = \pi - \cos^{-1} x$, $x \in [-1,1]$

ii. $\sec^{-1}(-x) = \pi - \sec^{-1}x$, $|x| \geq 1$

iii. $\cot^{-1}(-x) = \pi - \cot^{-1}x$, $x \in R$

Proof: $\cos^{-1}(-x) = \pi - \cos^{-1} x$, $x \in [-1,1]$
Let $\cos^{-1}(-x) = y$
$\cos y = -x$

$x = -\cos y$
$x = \cos(\pi - y)$
Since, $\cos(\pi - y) = -\cos y$
$\cos^{-1} x = \pi - y$
Hence, $\cos^{-1} x = \pi - \cos^{-1}(-x)$

Property 5

i. $\sin^{-1}x + \cos^{-1}x = \pi/2$, $x \in [-1,1]$

ii. $\tan^{-1}x + \cot^{-1}x = \pi/2$, $x \in R$

iii. $\operatorname{cosec}^{-1}x + \sec^{-1}x = \pi/2$, $|x| \geq 1$

Proof: $\sin^{-1}x + \cos^{-1}x = \pi/2, x \in [-1,1]$
Let $\sin^{-1} x = y$ or $x = \sin y = \cos(\frac{\pi}{2} - y)$

$\cos^{-1} x = \frac{\pi}{2} - y$

$\cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x$

Hence, $\cos^{-1} x + \sin^{-1} x = \frac{\pi}{2}, x \in [-1,1]$

Property 6

i. $\tan^{-1}x + \tan^{-1}y = \tan^{-1}(\frac{x+y}{1-xy}), xy < 1.$

ii. $\tan^{-1}x - \tan^{-1}y = \tan^{-1}(\frac{x-y}{1+xy}), xy > -1.$

Proof : $\tan^{-1}x + \tan^{-1}y = \tan^{-1}((x+y)/(1-xy)), xy < 1.$
Let $\tan^{-1} x = A$

And $\tan^{-1} y = B$
Then, $\tan A = x$
$\tan B = y$, where $A,B \in(-\frac{\pi}{2}, \frac{\pi}{2})$
Now, $\tan(A+B) = (\tan A + \tan B)/(1-\tan A \tan B)$
$\tan(A+B) = \frac{x+y}{1-xy}$
$A + B = \tan^{-1} \frac{x+y}{1-xy}$
$\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy}$

Property 7

i. $2\tan^{-1}x = \sin^{-1} (2x/(1+x^2)), |x| \leq 1$

ii. $2\tan^{-1}x = \cos^{-1}((1-x^2)/(1+x^2)), x \geq 0$

iii. $2\tan^{-1}x = \tan^{-1}(2x/(1 - x^2)), -1 < x <1$

Property 8

$i.\sin^{-1} x \pm \sin^{-1} y = \sin^{-1}\{(x\sqrt{1-y^2} \pm y\sqrt{1-x^2}$

ii. $\cos^{-1} x \pm \cos^{-1} y = \cos^{-1}\{ xy \mp \sqrt{1-x^2}\sqrt{1-y^2}$

Other trigonometric functions can be proved in the similar manner.

Exercise

Q1. Find the principal values of

i. $\sin^{-1}(-\frac{1}{2})$

ii. $\sec^{-1}(-2)$

iii. $\sin^{-1}(\sin\frac{5\pi}{6})$

iv. $\tan^{-1}(\tan\frac{2\pi}{3})$

v. $\tan(\frac{1}{2}\sin^{-1}\frac{3}{4})$

2. Prove $\cos^{-1}(\frac{5}{13}) = \tan^{-1}(\frac{12}{5})$

3. Find the value of $\tan^{-1}\sqrt{3} - \sec^{-1}(-2)$

4. Verify $\sin^{-1}\frac{\sqrt{2}}{2} - \sin^{-1}\frac{1}{2} = \frac{\pi}{12}$

5. Show that $\cos^{-1}(-x) = \pi - \cos^{-1}x$

6. Show that $2\tan^{-1}\frac{1}{2} = \tan^{-1}\frac{4}{3}$

7. Simplify: $\sin(2\cos^{-1}x)$
8. Prove the following
   i. $\tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{13} = \tan^{-1}\frac{2}{9}$

   ii. $\cot^{-1}7 + \cot^{-1}8 + \cot^{-1}18 = \cot^{-1}3$
   iii. $2\tan^{-1}\frac{3}{4} - \tan^{-1}\frac{17}{13} = \frac{\pi}{4}$
   iv. $\sin^{-1}(\frac{63}{65}) = \sin^{-1}(\frac{5}{13}) + \cos^{-1}(\frac{3}{5})$
9. If $\sin^{-1}\frac{2a}{1+a^2} - \cos^{-1}\frac{1-b^2}{1+b^2} = \tan^{-1}\frac{2x}{1-x^2}$, then prove that $x = \frac{a-b}{1+ab}$
10. Prove that $2\tan^{1}(\sqrt{\frac{a-b}{a+b}}\tan\frac{\theta}{2}) = \cos^{-1}\frac{a\cos\theta+b}{a+b\cos\theta}$
11. Write in the simplest form.
   i. $\sin^{-1}\{x\sqrt{1-x} - \sqrt{x}\sqrt{1-x^2}\}$

   Ii. $\cot^{-1}\frac{a}{\sqrt{x^2-a^2}}$, $|x|>a$

12. Solve the following for x:
   i. $\tan^{-1}2x + \tan^{-1}3x = n\pi + \frac{3\pi}{4}$

   ii. $\tan(\cos^{-1}x)\sin(\cot^{-1}\frac{1}{2})$

   iii. $\tan^{-1}\frac{x}{2} + \tan^{-1}\frac{x}{3} = \frac{\pi}{4}$, $0 < x < \sqrt{6}$

iv. $\cos(\tan^{-1} x) = \sin(\cot^{-1}\frac{3}{4})$

v. If $\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = \pi$, prove that $x^2 + y^2 + z^2 + 2xyz = 1$

A function f: D → R is said to be **continuous** at x =c if its both left continuous and right continuous at x = c ,i.e ..if

$\lim_{x\to c-} f(x) = \lim_{x\to c+} f(x)$= f(x)

Function f is **differentiable** if its graph has no corner points and has unique tangent (with finite slope) at every point on the curve.

A function f: D → R is said to be differentiable at x = c if it is both left differentiable and right differentiable at x = c , i.e., if both the limits

$\lim_{h\to 0}\frac{f(c-h)-f(c)}{-h}$ and $\lim_{h\to 0}\frac{f(c+h)-f(c)}{-h}$ exist finitely and are equal.

Example on continuity

Is the function f defined by f(x) = $\begin{cases} x \; if \; x < 1 \\ 5 \;\; if \; x \geq 1 \end{cases}$ continuous at x = 1

f(1) = 5

$\lim_{x\to 1+} [f(x)] = \lim_{x\to 1}[5] = 5$

$\lim_{x\to 1-} [f(x)] = \lim_{x\to 1-} [x]$

Putting x = 1- h , as x → 1, h→ 0

$\lim_{h\to 0}[1-h] = 1$

Since $\lim_{x\to 1+} [f(x)] \neq \lim_{x\to 1-} [f(x)]$

Hence, f(x) is not continuous at x = 1

Example on differentiability

Let f(x) =x |x|, for all x ϵ R . Discuss the differentiability of f(x) at x = 0.

Given f(x) =x |x| = $\begin{cases} x(-x) \; if \; x < 0 \\ x(x) \;\; if \; x \geq 0 \end{cases}$ = $\begin{cases} -x^2 if \; x < 0 \\ x^2 \; if \; x \geq 0 \end{cases}$

To check differentiability at x = o, put x = 0 + h as x→ 0, h→ 0

R f'(0) = $\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$

$= \lim_{h\to 0}[\frac{h^2-0}{h}]$

$= \lim_{h\to 0}[h]$

= 0

put x = 0 - h as x→ 0, h→ 0

L f'(0) = $\lim_{h\to 0}\frac{f(0-h)-f(0)}{-h}$

$= \lim_{h\to 0}[\frac{-h^2-0}{-h}]$

$= \lim_{h\to 0}[h]$

= 0

Since the left hand limit is equal to the right hand limit = 0, therefore f is differentiable at x = 0.

Exercise:

1. Check the continuity of the function f(x) = $2x^2 - 1$ at x=3

2. Show that the function f(x) = $\begin{cases} 3x-2 \; if \; 0 < x \leq 1 \\ 2x^2 - x \quad if \; 1 < x \leq 2 \; is \; continuous \; at \; x = 2 \\ 5x - 4 \; if \; x > 2 \end{cases}$

3. If f(x) = $\begin{cases} e^{1/x} \; if \; x \neq 0 \\ 1 \; if \; x = 0 \end{cases}$, find whether f (x) is continuous at x = 0.

4. Test the continuity of the function f(x) = $\begin{cases} \frac{|x|}{x} \; if \; x \neq 0 \\ 1 \;\; if \; x = 0 \end{cases}$, at the origin.

5. Discuss the continuity of f(x) = $\begin{cases} \frac{1}{2} - x \quad if \; 0 \leq x < \frac{1}{2} \\ 1 \;\; if \; x = \frac{1}{2} \\ \frac{3}{2} - x \; if \; \frac{1}{2} < x \leq 1 \end{cases}$ at x = 1/2

11. Show that f(x) = $|\cos x|$ is a continuous function.

12. Show that the function f(x) = $x^{1/3}$ is not differentiable at x = 0.

13. Show that f(x) = $\begin{cases} 12x - 13 \, , if \; x \leq 3 \\ 2x^2 + 5, \;\; if \;\; x > 3 \end{cases}$ is differentiable at x = 3. Also , find f '(3).

14. For what choice of a and b is the function f(x) = $\begin{cases} x^2, when \ x \leq c \\ ax + b \ , when \ x > c \end{cases}$, is differentiable at x =c .

15. Show that the function f(x) = $|x - 3|$ is continuous but not differentiable at x = 3.