GIRLS' HIGH SCHOOL AND COLLEGE

2020 – 2021

CLASS - 12 A&B

PHYSICS

WORKSHEET- 05

Chapter- GAUSS' THEOREM

Topic – INTRODUCTION TO GAUSS' THEOREM

INSTRUCTIONS: Parents kindly instruct your ward to visit the relevant websites or refer Nootan ISC 12 Physics- 12 by Kumar & Mittal (Nageen Prakashan) or Physics - 12 by DK Tyagi(Balaji Publications) to go though the topics- Area Vector, Solid Angle, Electric Flux and the theorem thoroughly to answer the following questions on the given topic.

NOTE: This topic is another important topic in Electrostatics and

needs a thorough understanding. Go through it twice thrice to understand the concept of area vector, solid angle and electric flux properly. You also need to revise Electric Field again as this topic is an extension of the concept of field along with electric lines of force.

You also need to go through scalar and vector quantities. Till now you had the concept that Area is a scalar quantity but now you will know that area basically is a vector quantity, as the way a surface is oriented with respect to any field decides the effect of the field on it.

Go through scalar and vector products properly. In this worksheet we are only going to introduce the theorem to you and its applications will be dealt with in the next worksheet.

Now go through the subject matter carefully.

Gauss' Theorem

Syllabus

Electric flux, Gauss' theorem in electrostatics and its applications to find field due to infinitely long straight wire, uniformly charged infinite plane sheet and uniformly charged thin spherical shell.

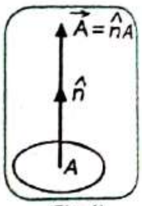
INTRODUCTION

German physicist Gauss had formulated a relation between charge distribution and electric field produced by charges. This relation is popularly known as Gauss' theorem and it is very useful in calculating electric field due to a symmetric charge distribution. In this chapter the concepts of electric flux, Gauss' law and its applications have been discussed.

1 Area Vector

A plane area, such as A in Fig. 1, can be regarded as possessing both magnitude and direction. Its magnitude is the amount of the area and its direction is the direction of the outward drawn normal to the plane of the area. Hence, the area A can be represented by a vector $\vec{\mathbf{A}}$ along the outward drawn normal to the area, the length of the vector representing the magnitude A. Thus,

$$\vec{\mathbf{A}} = \hat{\mathbf{n}}\, A$$



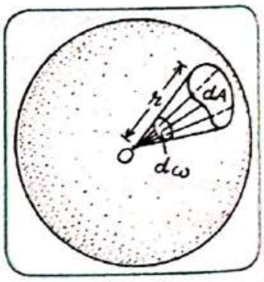
(Fig. 1)

is called the area vector $\vec{\mathbf{A}}$, for the surface under consideration. Here $\hat{\mathbf{n}}$ is a unit vector along the outward drawn normal.

2 Solid Angle

The arc of a circle subtends an angle at the centre of the circle. This angle is called a 'plane angle'. Its unit is 'radian' (rad). 1 ***radian is the angle which an arc of length equal to the radius of a circle subtends at the centre of the circle.***

Similarly, the area of a spherical surface subtends an angle at the centre of the sphere. This angle is called the 'solid angle' and is represented by 'ω'. Let O (Fig. 2) be the centre and r the radius of a sphere. Let dA be a small area element of the surface of the sphere. If the points situated on the boundary of this area be joined to O, then the lines so drawn will subtend a solid angle $d\omega$ at O. Since, the spherical area dA is directly proportional to the square of radius (r_2), the ratio dA/r_2 is a constant. This ratio is called the solid angle $d\omega$ subtended by the area dA at the centre O of the sphere. Thus,



(Fig. 2)

$$d\omega = dA / r^2. \quad \text{... (i)}$$

The unit of solid angle is 'steradian (sr)'. In eq. (i),

if $\quad dA = r^2, \quad$ then $\quad d\omega = 1.$

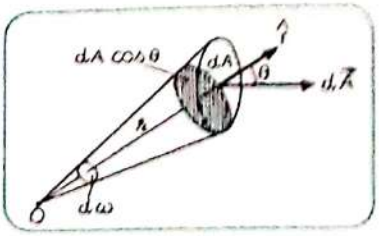
Therefore, 1 ***steradian is the solid angle subtended by a part of the surface of a sphere at the centre of the sphere, when the area of the part is equal to the square of the radius of the sphere.***

The entire surface area of a sphere is $A = 4\pi r^2$. Hence, the solid angle subtended by the entire surface of the sphere at its centre is

$$\omega = \frac{A}{r^2} = \frac{4\pi r^2}{r^2} = 4\pi \text{ steradian.}$$

In fact, the solid angle subtended by a *closed* surface of any shape at a point inside it is 4π steradian.

Now, let us consider an area element dA situated at a distance r from a point O (Fig. 3). Let $\hat{\mathbf{r}}$ be a unit vector along the line joining the point O to the area element dA. Let θ be the angle between $\hat{\mathbf{r}}$ and the vector $d\vec{A}$ representing the area element dA. If we draw a sphere with O as centre and r as radius, then the projection of area element dA on the surface of the sphere will be $dA \cos\theta$ (shown shaded). Therefore, the solid angle subtended by area element dA at the centre O of the sphere is



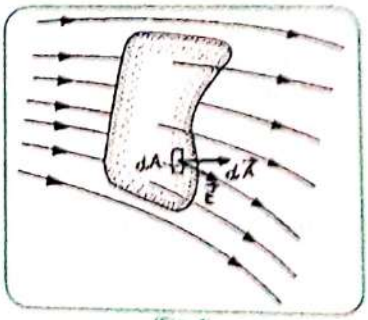
(Fig 3)

$$d\omega = \frac{dA\,\cos\theta}{r^2}$$

3 Electric Flux

The electric flux is a property of electric field. We know that an electric field can be visualized by lines of force; electric field is stronger where the lines of force are closer, and vice-versa. ***The electric flux is a measure of the number of lines of force passing through some surface held in the electric field.*** It is denoted by Φ_E.

Let there be an arbitrary surface immersed in an electric field (not necessarily uniform), as shown by lines of force (Fig. 4). Let us consider an imaginary 'small' surface element dA, the surface over the element being practically plane and the electric field uniform. The surface element may be represented by a vector $d\vec{A}$ of magnitude dA, directed along the outward drawn normal to the element. Let $\vec{E}$ be electric field at the location of the element $d\vec{A}$. Then, ***the scalar product $\vec{E}\cdot d\vec{A}$ is defined as 'electric flux' through the element.*** The electric flux through the entire surface is therefore

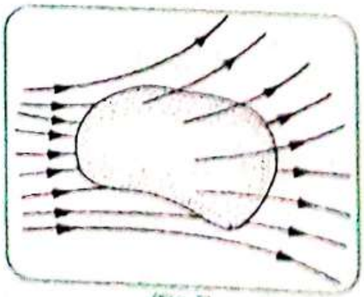


(Fig 4)

$$\Phi_E = \int_A \vec{E}\cdot d\vec{A}$$

where $\int_A$ is the (surface) integral over the entire surface. Φ_E is positive for the lines of force pointing outwards (leaving the surface); and negative for those pointing inwards (entering the surface). Thus, ***the electric flux linked with a surface in an electric field may be defined as the surface integral of the normal component of the electric field over that surface.***

In case of a 'closed' surface (Fig. 5), that is, a surface that completely encloses a volume (like the surface of a balloon), the net flux through the surface is given by


(Fig 5)

$$\Phi_E = \oint \vec{E}\cdot d\vec{A}.$$

The electric flux Φ_E, being the scalar product of two vectors, is a scalar.

Let us now consider a plane surface of area A placed in a ***uniform*** electric field $\vec{E}$. The normal to the surface makes an angle θ with the field. By definition, the electric flux through the surface is given by

$$\Phi_E = \oint_A \vec{E} \cdot d\vec{A}$$

where $d\vec{A}$ is the area vector of a small element dA on the surface. Since, $d\vec{A}$ is normal to the surface, the angle between vectors $\vec{E}$ and $d\vec{A}$ is θ and

$$\vec{E} \cdot d\vec{A} = E\, dA \cos\theta.$$

$$\therefore \quad \Phi_E = \int_A E\, dA \cos\theta = E\cos\theta \int_A dA.$$

But $\int_A dA = A$ (given).

$$\therefore \quad \Phi_E = E\, A \cos\theta.$$

Special Cases :

(i) If the plane surface is *normal* to the electric field $\vec{E}$ (Fig. 6), that is, $\theta = 0$, then

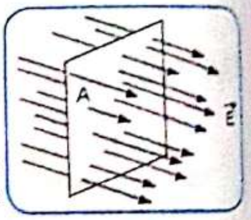
$$\Phi_E = E\,A \cos 0 = E\,A.$$

(ii) If the plane surface is parallel to the electric field ($\theta = 90°$), then

$$\Phi_E = E\,A \cos 90° = 0.$$

(iii) For field lines entering the plane surface normally ($\theta = 180°$), we have

$$\Phi_E = E\,A \cos 180° = -E\,A.$$



(Fig. 6)

Electric Flux Density : In an electric field, the ratio of electric flux Φ_E through a surface to the area A of the surface is called the 'electric flux density' at the location of the surface. That is

$$\text{electric flux density} = \frac{\Phi_E}{A}.$$

For a plane surface *normal* to the electric field, $\Phi_E = E\,A$. Thus,

$$\text{electric flux density} = \frac{E\,A}{A} = E.$$

The unit of electric flux density is same as that of electric field. Since, electric field is a vector quantity, the electric flux density is also a vector quantity (although electric flux is a scalar quantity).

Unit and Dimensions of Electric Flux : By definition

$$\Phi_E = EA\cos\theta.$$

Therefore, the SI unit of Φ_E is

$$\frac{\text{N}}{\text{C}} \times \text{m}^2 = \text{N-m}^2\text{C}^{-1}$$

E is also expressed in V/m.

Therefore, another SI unit of Φ_E is $\frac{\text{V}}{\text{m}} \times \text{m}^2 = \text{V-m}.$

The dimensions of Φ_E,

$$\text{dimensions of } E \times \text{dimensions of } A = \left[\frac{\text{M L T}^{-2}}{\text{A T}}\right] \times [\text{L}^2] = [\text{M L}^3\text{T}^{-3}\text{A}^{-1}].$$

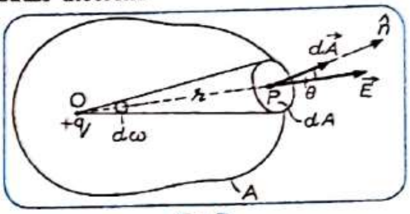
4 Gauss' Theorem

The Gauss' theorem in electrostatics gives a relation between the electric flux through any closed hypothetical surface (called a Gaussian surface) and the charge enclosed by the surface. It states that *the electric flux Φ_E through any closed surface is equal to $1/\varepsilon_0$ times the 'net' charge q enclosed by the surface.* That is,

$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = \frac{q}{\varepsilon_0}$$

where ε_0 is permittivity of free space. This is the *integral* form of Gauss' theorem.

Proof: Let us consider a point-charge $+q$ situated at O inside a closed surface A (Fig. 7). Let dA be a small area element surrounding a point P on the surface. Let $OP = r$. The area element may be represented by a vector $d\vec{A}$ drawn outward along the normal to the element.



(Fig. 7)

Let $\vec{E}$ be the electric field intensity at P due to charge $+q$ at O. Its direction is along OP. The total normal electric flux through the area element dA is

$$d\Phi_E = \vec{E} \cdot d\vec{A} = E\, dA \cos\theta$$

where θ is the angle between the vectors $\vec{E}$ and $d\vec{A}$.

Now, $E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}$.

$$\therefore \quad d\Phi_E = \frac{q}{4\pi\varepsilon_0}\frac{dA\cos\theta}{r^2}.$$

But $\frac{dA\cos\theta}{r^2}$ is the solid angle $d\omega$ subtended by the area dA at the point O. Therefore,

$$d\Phi_E = \frac{q}{4\pi\varepsilon_0}\, d\omega.$$

The total normal electric flux Φ_E through the entire surface A is

$$\Phi_E = \frac{q}{4\pi\varepsilon_0}\oint d\omega.$$

Now, $\oint d\omega = 4\pi$, the solid angle subtended by the entire closed surface A at the point O.

$$\therefore \quad \Phi_E = \frac{q}{\varepsilon_0}.$$

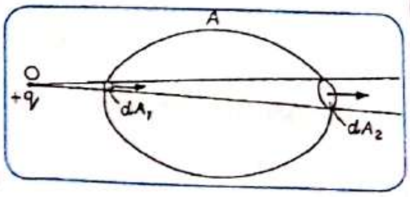
This is what we had to prove.

Note

(i) If there are several charges $+q_1, +q_2, +q_3, -q_4, -q_5, \ldots$ etc., the same argument can be applied to each in turn. Hence, the total flux through the surface due to all of them is

$$\frac{1}{\varepsilon_0}(q_1 + q_2 + q_3 - q_4 - q_5 + \ldots) = \frac{1}{\varepsilon_0}\Sigma q$$

where Σq is the algebraic sum of all the charges enclosed by the closed surface.



(ii) If the charge q be outside the closed surface A (figure), the total flux through the surface is zero because the cone with vertex at q cuts off area dA_1 where it enters, and area dA_2 where it leaves the surface. The flux through dA_1 is $-\frac{q}{4\pi\varepsilon_0}d\omega$ (*inward*) and that through dA_2 is

$+\frac{q}{4\pi\varepsilon_0} d\omega$ (*outward*). Hence, the total flux through the two areas is zero. This is true for two areas cut-off by any such cone. Hence, for the whole closed surface the total flux is zero.

(iii) Gauss' theorem is valid for a closed surface, no matter what is its shape or size.

(iv) The electric flux passing through a closed surface depends upon the net charge enclosed by the surface. It does not depend upon the position of charge or distribution of charge inside the surface.

(v) The electric flux passing through a closed surface, apart from the net charge enclosed depends upon the dielectric constant (K) of the medium and $\Phi_E \propto \frac{1}{K}$.

(vi) The net electric flux through a closed surface due to the charge outside the surface is zero.

(vii) The charge q in the Gauss' theorem is the sum of all the charges located anywhere inside the closed surface.

(viii) Gauss' theorem is mainly used for symmetrical charge distribution.

(ix) Gauss' theorem and Coulomb's law are complementary to each other.

5 Gaussian Surface and its Properties

A Gaussian surface is an arbitrary closed surface in three dimensional space through which flux of vector field (the gravitational field, the electric field or magnetic field) is calculated.

The examples of some valid Gaussian surfaces include the surface of sphere, cylinder, cube etc. Some surfaces cannot be used as Gaussian surfaces such as disc surface, square surface etc.

Essential properties of Gaussian surface are :

(i) It should be a closed surface so that a clear distinction can be made between points that are inside the surface, on the surface and outside the surface.

(ii) This surface must pass through the point where electric field is to be calculated.

(iii) The surface must have a shape according to the symmetry of the source, so that the field is normal to the surface at each point and constant in magnitude.

(iv) For a system of charges the Gaussian surface should not pass through any discrete charge. It is because electric field at the location of any charge is not well defined. However, the Gaussian surface can pass through a continuous charge distribution.

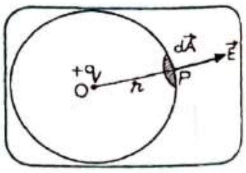
6 Applications of Gauss' Theorem

Gauss' theorem is very useful in computing the electric fields due to a system of charges or symmetrical continuous distribution of charge.

(i) Electric Field due to a Point Charge : Deduction of Coulomb's Law from Gauss' Theorem

Let us consider a point charge $+q$ placed at O [Fig. (8)]. The electric field due to this point charge is to be calculated at P. The magnitude of the electric field is the same at all points lying at the same distance from O. To find the electric field ($\vec{E}$) at the point P distant r from O, draw a spherical surface taking O as centre and radius r as the Gaussian surface. Therefore, electric flux

$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = \oint E\,dA \cos 0 = EA$$



(Fig. 8)

But $A = 4\pi r^2$ = surface area of the spherical Gaussian surface

$$\Phi_E = E \cdot 4\pi r^2 \quad \ldots(i)$$

But from Gauss' theorem, $\Phi_E = \frac{q}{\varepsilon_0}$ ...(ii)

From eqs. (i) and (ii)

$$E \cdot 4\pi r^2 = \frac{q}{\varepsilon_0}$$

or

$$E = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2}$$

Force acting on a test charge q_0 placed at P,

$$F = q_0 E = \frac{1}{4\pi\varepsilon_0} \frac{q q_0}{r^2}$$

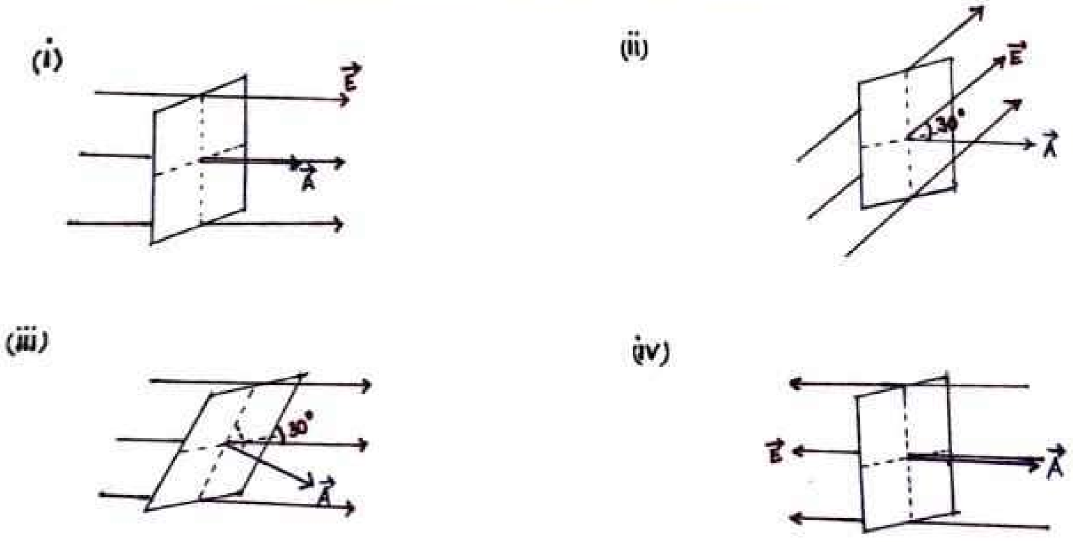
This represents Coulomb's law.

Now answer the following questions :-

Q1) Explain the term Area Vector.

Q2) What do you mean by positive and negative Area Vector?

Q3) For the following figures state the angle made by area vector with electric field direction :-



Q4) Define 1 steradian and give its dimensional formula.

Q5) Show that the entire surface area of a sphere subtends a solid angle of 4π at the centre.

Q6) Show that the small solid angle $d\omega$ subtended by any small surface area dA at any point such that it makes an angle θ with line joining point and area element is $d\omega = \frac{dA\cos\theta}{r^2}$.

Q7) Define Electric Flux. What kind of a quantity is it?

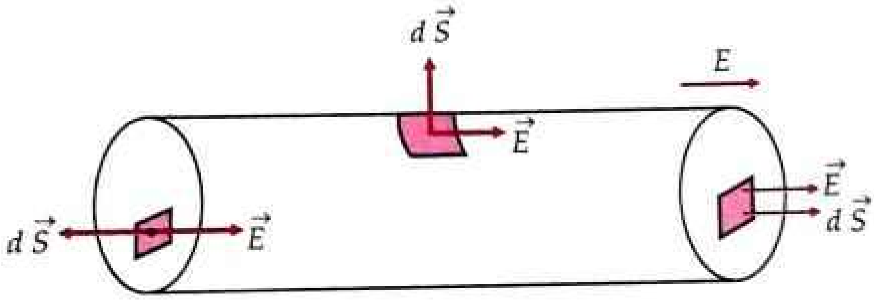
Q8) Give the mathematical expression for Flux, its unit and dimensional formula.

Q9) How does electric flux depend upon angle θ that the surface makes with the field? When is flux maximum (for what θ)?

Q10) A circular plane sheet of radius 10 cm is placed in a uniform electric field of $5 \times 10^{5}\ NC^{-1}$, making an angle of 60° with the field. Calculate the electric flux through the sheet.

Q11) If $\vec{E} = 6\hat{i} + 3\hat{j} + 4\hat{k}$, calculate the electric flux through a surface of area 20 units in Y- Z plane.

Q12) A cylinder is placed in a uniform electric field $\vec{E}$ with its axis parallel to the field. Show that the total electric flux through the cylinder is zero.



Q13) State and prove Gauss' Theorem.

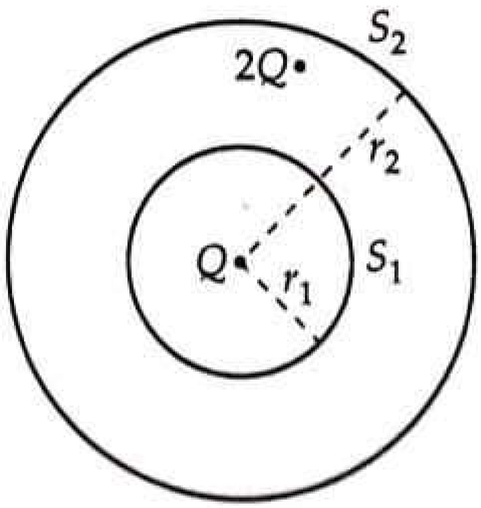
Q14) Calculate the number of electric lines of force originating from a charge of 1 C.

Q15) A point charge produces an electric flux of $-1.0 \times 10^{3}\ N\,m^{2}\,C^{-1}$ which passes through a Gaussian sphere of radius 10 cm centered on the charge. Compute the point- charge. If the radius of the sphere be doubled, how much flux would pass through its surface?

$(\varepsilon_0 = 8.85 \times 10^{-12}\ C^{2}\,N^{-1}\,m^{-2})$

Q16) A sphere S_1 encloses charge '*q*'. There is a larger concentric sphere S_2, with no additional charge between S_1 and S_2.Find the ratio of electric flux through S_1 and S_2.

Q17) S_1 and S_2 are two concentric spheres (S_2 outer and S_1 inner) enclosing charges Q and 2Q respectively.



(i) What is the ratio of electric flux through S_1 and S_2?

(ii) How will the electric flux through S_1 change, if a medium of dielectric constant ‘k’ is introduced in the space inside S_1 in place of air?

(iii) How will the electric flux through sphere S_1 change, if a medium of dielectric constant ‘k’ is introduced in space between S_1 and S_2 in place of air?

Q18) What can be a possible use of Gauss’ Theorem?

Q19) What is a Gaussian Surface? Give its properties.

Q20) Deduce Coulomb’s Law using Gauss’ Theorem.

END