GIRLS' HIGH SCHOOL AND COLLEGE,PRAYAGRAJ

2020 – 2021

CLASS - 12 B & C

MATHEMATICS

WORKSHEET NO. 5

CHAPTER: MATRICES and DETERMINANTS

Note: Parents are expected to ensure that the student spends two days to read and understand the topic according to the book or the website referred and thereafter answer the questions.

Book : ISc mathematics for class 12 by OP Malhotra

Website: www.khanacademy.org ,www.topperlearning.com or any other relevant website.

Exercise

iUsing the properties of the determinants , prove that

(i) $\begin{vmatrix} 1+a_1 & a_2 & a_3 \\ a_1 & 1+a_2 & a_3 \\ a_1 & a_2 & 1+a_3 \end{vmatrix} = 1 + a_1 + a_2 + a_3$

(ii) $\begin{vmatrix} a & b-c & c-b \\ a-c & b & c-a \\ a-b & b-a & c \end{vmatrix} = (a + b - c)(b + c - a)(c + a - b)$

(iii) $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a + b +c)^3$

(iv) $\begin{vmatrix} 1 & a & a^2 \\ a^2 & 1 & a \\ a & a^2 & 1 \end{vmatrix} = (a^3 -1)^2$

(v) $\begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} = (5x + 4)(4 - x)^2$

A square matrix A is said to be **singular** if det $[A] = 0$, otherwise it is said to be **non singular.**

Adjoint and Inverse of a Matrix

The adjoint of a square matrix is the transpose of the matrix obtained by replacing each element of A by its cofactor in $|A|$

Theorem : Let A be a square matrix of order n then $A(\text{adj}A) = |A|I_n = (\text{adj}A) A$

Ex: Let A be a square matrix of order 3×3

A = $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -3 \\ -1 & 2 & 3 \end{bmatrix}$, Find it's adjoint.

Let A_{ij} be the cofactors of a_{ij} in A . Then , the cofactors of elements of A are given by

$A_{11} = \begin{vmatrix} 1 & -3 \\ 2 & 3 \end{vmatrix} = 9$

$A_{12} = -\begin{vmatrix} 2 & -3 \\ -1 & 3 \end{vmatrix} = -3$

$A_{13} = \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} = 5$

$A_{21} = -\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = -1$

$A_{22} = \begin{vmatrix} 1 & 1 \\ -1 & 3 \end{vmatrix} = 4$

$A_{23} = -\begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} = -3$

$A_{31} = \begin{vmatrix} 1 & 1 \\ 1 & -3 \end{vmatrix} = -4$

$A_{32} = -\begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} = 5$

$A_{33} = \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = -1$

$$\text{Adj } A = \begin{bmatrix} 9 & -3 & 5 \\ -1 & 4 & -3 \\ -4 & 5 & -1 \end{bmatrix}^t = \begin{bmatrix} 9 & -1 & -4 \\ -3 & 4 & 5 \\ 5 & -3 & 1 \end{bmatrix}$$

Exercise:

Q. Find the adjoint of the matrices.

1. $\begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix}$

2. $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

3. $\begin{bmatrix} -1 & 0 & 2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$

4. $\begin{bmatrix} -1 & -2 & 3 \\ -2 & 1 & 1 \\ -4 & -5 & 2 \end{bmatrix}$

5. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Q. If A = $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 1 & 0 & 3 \end{bmatrix}$, find the value iof A (Adj A) without finding Adj A.

Hint : A (AdjA) = |A |I

Q. If A = $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ and B = $\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$, prove that adjAB = (adjA)(adjB)

Q. Prove that |adjAB| = |adjA||adjB|

Q.For the matrix A = $\begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 18 & 2 & 10 \end{bmatrix}$, show that A (adjA) = 0

Inverse of a matrix.

A square matrix of order n is invertible if there exists a square matrix B of the same order such that AB = I_n = BA. In such case, we say that inverse of A is B and is written as A^{-1} = B.

Theorem 1: Every invertible matrix possesses a unique inverse.

PROOF Let A be an invertible matrix of order n x n. Let B and C be two inverses of A. Then,

AB = BA = I_n ------------(i)

and AC = CA = I_n ------------(ii)

Now, AB = I_n

⇒ C(AB) = C I_n [Pre-multiplying both sides by C)

⇒(CA) B = C I_n [By associativity of multiplication]

⇒ CA = I_n from (ii)]

⇒I_n B = C I_n

⇒ B = C Hence, an invertible matrix possesses a unique inverse.

Theorem 2 : A square matrix is invertible iff it is non-singular.

PROOF: Let A be an invertible matrix. Then, there exists a matrix B such that

AB = I_n = BA

| AB|= | I_n |

|A|| B| = 1

|A| ≠ 0

⇒A is a non-singular matrix.

Conversely, let A be a non-singular square matrix of order n. Then,

A (adj A) = | A| I_n, = (adj A) A

$A\left(\frac{\text{adj}A}{|A|}\right) = I_n = \left(\frac{\text{adj}A}{|A|}\right)A$ since it is a non singular matrix ,therefore |A| ≠ 0

$\Rightarrow A^{-1} = \left(\frac{\text{adj}A}{|A|}\right)$

This is the formula to find the inverse of a non –singular square matrix A.

Thus $\mathbf{A^{-1} = \frac{1}{|A|} adjA}$

Theorem: Let A,B,C be square matrices of the same order n. If A is a non-singular matrix , then

i. AB = AC $\Rightarrow$ B = C

ii. BA = CA $\Rightarrow$ B = C

Theorem : If A and B are invertible matrices of the same order , then $(AB)^{-1} = B^{-1}A^{-1}$

Ex. A = $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -3 \\ -1 & 2 & 3 \end{bmatrix}$

Adj A = $\begin{bmatrix} 9 & -3 & 5 \\ -1 & 4 & -3 \\ -4 & 5 & -1 \end{bmatrix}^t$ = $\begin{bmatrix} 9 & -1 & -4 \\ -3 & 4 & 5 \\ 5 & -3 & 1 \end{bmatrix}$

$|A| = 1\begin{vmatrix} 1 & -3 \\ 2 & 3 \end{vmatrix} -1\begin{vmatrix} 2 & -3 \\ -1 & 3 \end{vmatrix} +1\begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix}$

= 1(3 + 6) -1(6 – 3) +1(4 + 1)

= 9 – 3 + 5 = 11

Since | A| $\neq$ 0 therefore the inverse of the matrix exists.

$A^{-1} = \frac{1}{11}\begin{bmatrix} 9 & -1 & -4 \\ -3 & 4 & 5 \\ 5 & -3 & 1 \end{bmatrix}$

Exercise:

Q. Find the inverse of each of the following matices:

1. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

2. $\begin{bmatrix} a & b \\ c & \frac{1+bc}{a} \end{bmatrix}$

3. $\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & \sin\theta & -cos\theta \end{bmatrix}$

Q. Find the inverse of the given matrice $\begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix}$

and verify that $A^{-1} A = I_3$.

Q. If A = $\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$ and B = $\begin{bmatrix} 4 & 5 \\ 3 & 4 \end{bmatrix}$ verify that $(AB)^{-1} = B^{-1}A^{-1}$

Q. Given that A = $\begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$, compute A^{-1} and hence show that $2A^{-1} = 9I - A$

Q. If $A^{-1} = \frac{1}{10}\begin{bmatrix} 10 & -10 & 2 \\ 0 & 5 & -4 \\ 0 & 0 & 2 \end{bmatrix}$ and B = $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}$, find $(BA)^{-1}$

Q. If A = $\begin{bmatrix} 2 & -3 \\ 4 & 6 \end{bmatrix}$, verify that $(\text{adj } A)^{-1} = \text{adj}(A^{-1})$

Q. If A = $\begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$, find x and y such that $A^2 - xA + yI = 0$

Application of the matrices to the solution of linear equation (Martin's Rule)

Consider two linear equations

$a_1x + b_1y = c_1$

$a_2x + b_2y = c_2$

these can be written in the matrix form as

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Or AX = B

Or, $X = A^{-1} B$

(i) If $|A| \neq 0$ the system is consistent and has a unique solution .To obtain the solution compute A^{-1} by using $A^{-1} = \left(\frac{adjA}{|A|}\right)$ and use the formula $X = A^{-1}B$

(ii) If A = 0, the system of equations has either no solution or an infinite number of solutions.

(iii) Find (adj A)B.
(a) if (adj A) B $\neq$ 0, the system has no solution and is, therefore, inconsistent.
(b) If (adj A) B= 0, the system is consistent and has infinitely many solutions. In this case we say that the equations are dependent equations.

(iv) Same method is applicable for 3 × 3 square matrix.

Ex:Solve the following system of equations, using matrix method or Martin's Rule.

x + 2y+ z = 7

x + 3z = 11

2x -3y = 1

Or, $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 11 \\ 1 \end{bmatrix}$

Or, AX = B where A = , $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}$, X = $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and B = $\begin{bmatrix} 7 \\ 11 \\ 1 \end{bmatrix}$

$|A| = \begin{vmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ 2 & -3 & 0 \end{vmatrix}$ = 1(0 + 9) -2 (0 - 6) +1(-3 -0) = 9 + 12 – 3= 18 ≠0

$|A| \neq 0$

So the given system of equations has a unique solution given by $X = A^{-1} B$

$A_{11} = (-1)^{1+1} \begin{vmatrix} 0 & 3 \\ -3 & 0 \end{vmatrix} = 9$

$A_{12} = (-1)^{1+2} \begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix} = -6$

$A_{13} = (-1)^{1+3} \begin{vmatrix} 1 & 0 \\ 2 & -3 \end{vmatrix} = -3$

$A_{21} = (-1)^{2+1} \begin{vmatrix} 2 & 1 \\ -3 & 0 \end{vmatrix} = -3$

$A_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} = -2$

$A_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 2 \\ 2 & -3 \end{vmatrix} = 7$

$A_{31} = (-1)^{3+1} \begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix} = 6$

$A_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = -2$

$A_{33} = (-1)^{3+3} \begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} = -2$

$\text{adj}A = \begin{bmatrix} 9 & 6 & -3 \\ -3 & -2 & 7 \\ 6 & -2 & -2 \end{bmatrix}^t = \begin{bmatrix} 9 & -3 & 6 \\ 6 & -2 & -2 \\ -3 & 7 & -2 \end{bmatrix}$

$A^{-1} = \frac{1}{18} \begin{bmatrix} 9 & -3 & 6 \\ 6 & -2 & -2 \\ -3 & 7 & -2 \end{bmatrix}$

$X = A^{-1} B$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{18}\begin{bmatrix} 9 & -3 & 6 \\ 6 & -2 & -2 \\ -3 & 7 & -2 \end{bmatrix}\begin{bmatrix} 7 \\ 11 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{18}\begin{bmatrix} 63 & -33 & 6 \\ 42 & -22 & -2 \\ -21 & 77 & -2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{18}\begin{bmatrix} 36 \\ 18 \\ 54 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

⇒ x = 2, y =1 and z =3 is the required solution.

Ex : Show that the following system of equations is consistent.

2x – y + 3z= 5

3x + 2y –z = 7

4x + 5y -5z = 9

$$\text{Or, } \begin{bmatrix} 2 & -1 & 3 \\ 3 & 2 & -1 \\ 4 & 5 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}$$

$$\text{Or, } AX = B \text{ where } A = , \begin{bmatrix} 2 & -1 & 3 \\ 3 & 2 & -1 \\ 4 & 5 & -5 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & -1 & 3 \\ 3 & 2 & -1 \\ 4 & 5 & -5 \end{vmatrix} = 2(-10 + 5) + 1(-15 + 4) + 3(15 - 8) = 0$$

Since |A| = 0 hence the given matrix is singular, thus the given set of equations is inconsistent or consistent with infinite many solutions according as (adjA)B = 0 or (adjA)B ≠0. Taking out the cofactors of each element the adjoint of A is

$$\text{adjA} = \begin{bmatrix} -5 & 11 & 7 \\ 10 & -22 & -14 \\ -5 & 11 & 7 \end{bmatrix}^t = \begin{bmatrix} -5 & 10 & -5 \\ 11 & -22 & 11 \\ 7 & -14 & 7 \end{bmatrix}$$

$$(\text{adjA})B = \begin{bmatrix} -5 & 10 & -5 \\ 11 & -22 & 11 \\ 7 & -14 & 7 \end{bmatrix}\begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix} = \begin{bmatrix} -25 & 70 & -45 \\ 55 & -154 & -99 \\ 35 & -98 & 63 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0$$

Thus AX = B has infinitely many solutions .<u>To find these solutions , we put z= k in the first two equations and write them as follows</u>:

2x – y = 5 - 3k and 3x + 2y = 7 + k

Or,

$$\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5-3k \\ 7+k \end{bmatrix}$$

Or, AX = B where A = , $\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$, X = $\begin{bmatrix} x \\ y \end{bmatrix}$ and B = $\begin{bmatrix} 5-3k \\ 7+k \end{bmatrix}$

$|A| = \begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} = 4 + 3 = 7 \neq 0$ and adjA = $\begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}$

Therefore, $A^{-1} = \frac{1}{7}\begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}$

Now $X = A^{-1}B$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 5-3k \\ 7+k \end{bmatrix}$$

$$= \frac{1}{7}\begin{bmatrix} 10-6k+7+k \\ -15+9k+2k+14 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{17-5k}{7} \\ \frac{11k-1}{7} \end{bmatrix}$$

$x = \frac{17-5k}{7}$, $y = \frac{11k-1}{7}$ these values of x, and z = k also satisfy the third equation. Hence,

$x = \frac{17-5k}{7}$, $y = \frac{11k-1}{7}$ and z = k , where k is any real number satisfy the given system of equations .

How to find the solution for homogeneous system of linear equations

The equation for this system is given as AX = 0

If $|A| \neq 0$, the matrix is non singular and hence the homogeneous equation will have trivial solutions i.e. x = y = z = 0 .

If $|A| = 0$,then (adjA)B = 0 = (adjA) 0 = 0 so the given system of equations is always satisfied and it has infinite many solutions which can be obtained by giving any real value to one of the variables and then solving the remaining equations by matrix method (as solved in the above example).

Exercise

Q. Solve the following system of equations by matrix method:

1. 2x -3y = 1 , 3x -2y = 4

2. x + y =5 , z + y = 7 , z + x = 6

3. $\frac{2}{x}$ + $\frac{3}{y}$ + $\frac{10}{z}$ = 4

$\frac{4}{x}$ - $\frac{6}{y}$ + $\frac{5}{z}$ = 1

$\frac{6}{x}$ + $\frac{9}{y}$ - $\frac{20}{z}$ = 2

4. If A = , $\begin{bmatrix} 8 & -4 & 1 \\ 10 & 0 & 6 \\ 8 & 1 & 6 \end{bmatrix}$, Find A^{-1} . Solve the following system of linear equations:

8x – 4y + z =5, 10x + 6z = 4, 8x + y + 6z = $\frac{5}{2}$

5. Find k so that the system of equation may have unique solutions.

3x -2y + 2z = 1 , 2x + y + 3z = -1 , x – 3y + kz = 0

6. 3x + y -2z = 0 , x + y +z = 0, x -2y + z = 0

7. x + y – z = 0

x – 2y + z = 0

3x + 6y – 5z = 0
