GIRLS' HIGH SCHOOL AND COLLEGEs

2020 – 2021

CLASS - 12 B & C

MATHEMATICS

WORKSHEET NO. 4

CHAPTER: MATRICES and DETERMINANTS

Note: Parents are expected to ensure that the student spends two days to read and understand the topic according to the book or the website referred and thereafter answer the questions.

Book : ISc mathematics for class 12 by OP Malhotra

Website: www.khanacademy.org ,www.topperlearning.com or any other relevant website.

ELEMENTARY OPERATIONS (Transformations) OF A MATRIX

There are six operations on a matrix, three of which are due to rows and three due to columns. These operations are known as elementary operations or transformations.

i. The interchange of any two columns or rows of a matrix is denoted as $C_i \leftrightarrow C_j$ or $R_i \leftrightarrow R_j$

let $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ applying $C_1 \leftrightarrow C_2$

$A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$

ii. The multiplication of the elements of any column or row by a non-zero number. Let k be any non-zero constant multiplied to any ith column is denoted by $C_i \rightarrow k C_i$ similarly , the multiplication of the elements of the ith column by k , where $k \neq 0$ is denoted by $R_i \rightarrow kR_i$

Let $A = \begin{bmatrix} 3 & 6 \\ 0 & 9 \end{bmatrix}$

Then applying $\mathbf{C_2 \rightarrow 5\, C_2}$

$A = \begin{bmatrix} 3 & \mathbf{30} \\ 0 & \mathbf{45} \end{bmatrix}$

iii. The addition of any column or row , the corresponding elements of any other row or column multiplied by any non-zero number. Symbolically, the addition to the elements of the ith column , the corresponding elements to jth column multiplied by k is denoted by $C_i \rightarrow C_i + kC_j$, similarly for the operation in row is denoted by $R_i \rightarrow R_i + kR_j$

In matrix $B = \begin{bmatrix} 2 & -2 & 1 \\ \mathbf{1} & \mathbf{3} & \mathbf{4} \\ 0 & 2 & -2 \end{bmatrix}$

applying $R_2 \rightarrow R_2 - 2\, R_1$

{R_2 means (1 3 4) and $2R_1$ is (4 -4 2) therefore $R_2 - 2R_1$ is (-3 7 2)}

$$B = \begin{bmatrix} 2 & -2 & 1 \\ \mathbf{-3} & \mathbf{7} & \mathbf{2} \\ 0 & 2 & -2 \end{bmatrix}$$

INVERTIBLE MATRIX

A square matrix A of order m is said to be invertible , if there exist another square matrix B of same order m , such that AB = I = BA where I is the unit matrix of same order **.** The matrix B is called the inverse of matrix A and is denoted by A^{-1}

e.g. $A = \begin{bmatrix} 3 & 10 \\ 2 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & -10 \\ -2 & 3 \end{bmatrix}$ be two square matrices whose order is 2.

$$AB = \begin{bmatrix} 21-20 & -30+30 \\ 14-14 & -20+21 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$BA = \begin{bmatrix} 21-20 & 70-70 \\ -6+6 & -20+21 \end{bmatrix}$$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ which is of order 2

Thus AB = BA = I .Hence, A is invertible and $A^{-1} = B$

If B is the inverse of matrix A then A is also inverse of matrix B.

A rectangular matrix does not possess inverse matrix.

Theorem 1: Inverse of a square matrix, if it exist is unique .

Theorem 2: If A and B are invertible matrices of the same order then $(AB)^{-1} = B^{-1}A^{-1}$

How to find the inverse of a square matrix by elementary operations :

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ be a square matrix of order 3 , to find the inverse of matrix A , first we write the given matrix in the form A= IA , where I is the identity matrix of the same order of A .

Now apply a sequence of elementary row operations **or** column operations on A of LHS to convert it into identity matrix and same operations applied on I of RHS of A = IA (apply either row or column operation, both the operations will not take place together)

From the above step we get the new matrix equation of the type I = BA or I = AB where B is called the inverse of A.

Note: If applying one or more than one elementary row (or column) operations we obtained all zeros in one or more rows (or columns) of LHS of A = IA , then A is not invertible.

Ex .If A = $\begin{bmatrix} 1 & -1 & 0 \\ 2 & 5 & 3 \\ 0 & 2 & 1 \end{bmatrix}$, find A^{-1} using elementary operations.

Given A = I_3A

$$\begin{bmatrix} 1 & -1 & 0 \\ 2 & 5 & 3 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Operate $R_2 \to R_2 - 2R_1$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 7 & 3 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Operate $R_2 \to R_2 - 3R_3$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} A$$

Operate $R_3 \to R_3 - 2R_2$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & -3 \\ 4 & -2 & 7 \end{bmatrix} A$$

Operate $R_1 \to R_1 + R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & -3 \\ -2 & 1 & -3 \\ 4 & -2 & 7 \end{bmatrix} A$$

I_3 = BA where B = $\begin{bmatrix} -1 & 1 & -3 \\ -2 & 1 & -3 \\ 4 & -2 & 7 \end{bmatrix}$

Find the inverse of the following matrices, if it exists, using elementary operations:

1. $\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$
2. $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$
3. $\begin{bmatrix} -1 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$
4. $\begin{bmatrix} 2 & -1 & 4 \\ 4 & 0 & 2 \\ 3 & -2 & 7 \end{bmatrix}$
5. $\begin{bmatrix} 2 & -1 & 3 \\ -5 & 3 & 1 \\ -3 & 2 & 3 \end{bmatrix}$

6. $\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

7. $\begin{bmatrix} 1 & 1 \\ 2 & 5 \end{bmatrix}$

8. $\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$

9. $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

10. $\begin{bmatrix} 3 & 10 \\ 2 & 7 \end{bmatrix}$

11. $\begin{bmatrix} 10 & -2 \\ -5 & 1 \end{bmatrix}$

DETERMINANTS

Every square matrix can be associated to an expression or a number which is known as its determinant. If A = [aij] is a square matrix of order n, then the determinant is given by detA or, $|A|$

or, $$\begin{vmatrix} a_{11} & a_{12}\dots\dots\dots\dots a_{1n} \\ a_{21} & a_{22}\dots\dots\dots\dots a_{2n} \\ a_{n1} & a_{n2}\dots\dots\dots\dots a_{nn} \end{vmatrix}$$

If A =$[a_{11}]$ is a square matrix of order one then its determinant is given as $|a_{11}|$ = a_{11}

If A = $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$is a matrix of order two then its determinant is given as $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ = $a_{11}a_{22}$ - $a_{12}a_{21}$

For ex: $\begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix}$ = 3×6 – 4×5 = 18 – 20 = -2

Determinant of matrix of order 3

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

= a$\begin{vmatrix} e & f \\ h & i \end{vmatrix}$ –b $\begin{vmatrix} d & f \\ g & i \end{vmatrix}$ + c $\begin{vmatrix} d & e \\ g & h \end{vmatrix}$

= a(ei – fh) – b (di- fg) + c (dh – eg)

In the same way determinant of order 3 can be taken out by expanding column

Minor

The minor M_{ij} of an element a_{ij} is the value of the determinant obtained by deleting the ith row and jth column of the given determinant.

$\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix}$ in the given determinant, minor of 2(i.e M_{11}) is 5, minor of 3(M_{12}) is 4, minor of 4 (M_{21})is 3 and minor of 5(M_{22}) is 2

Cofactors of an element of a given determinant is given as :$A_{ij} = (-1)^{i+j} M\, ij$

Let the determinant be $\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix}$

Then the cofactors of each element is given as

$A_{11} = (-1)^{1+1} M_{11}$

$= (-1)^2\, 5$

$= 5$

$A_{12} = (-1)^{1+2} M_{12}$

$=(-1)^3 4$

$=-4$

$A_{21} = (-1)^{2+1} M_{21}$

$= (-1)^3 3$

$= -3$

$A_{22} = (-1)^{2+2} M_{22}$

$= 2$

In the same manner cofactors of determinant of order 3 can be taken out

Let the determinant be $\begin{vmatrix} 1 & -1 & 0 \\ 2 & 5 & 3 \\ 0 & 2 & 1 \end{vmatrix}$

$A_{11} = (-1)^{1+1}\, 1 \begin{vmatrix} 5 & 3 \\ 2 & 1 \end{vmatrix} =1\,(5 - 6)= -1$

$A_{12} = (-1)^{1+2}\, (-1) \begin{vmatrix} 2 & 3 \\ 0 & 1 \end{vmatrix} =(1)\,(2 - 0)= 2$

$A_{13} = (-1)^{1+3}\, 0 \begin{vmatrix} 2 & 5 \\ 0 & 2 \end{vmatrix} =0\,(4 - 0)= 0$

$A_{21}= (-1)^{2+1}\, 2 \begin{vmatrix} -1 & 0 \\ 2 & 1 \end{vmatrix} =-2\,(-1 - 0)= 2$

A_{22} = (-1) $^{2+2}$5 $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$ =5(1– 0)= 5

A_{23}= (-1) $^{2+3}$3 $\begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix}$ =-3(2– 0)= -6

A_{31} = (-1) $^{3+1}$ 0 $\begin{vmatrix} -1 & 0 \\ 5 & 3 \end{vmatrix}$ =0 (-3 – 0)= 0

A_{32} = (-1) $^{3+2}$ 2 $\begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix}$ =-2 (3 – 0)= -6

A_{33} = (-1) $^{3+3}$ 1 $\begin{vmatrix} 1 & -1 \\ 2 & 5 \end{vmatrix}$ =1 (5 + 2)= 7

Evaluate the following determinants from(Q1 to Q3)

1. $|-12|$
2. $\begin{vmatrix} 3 & 5 \\ -2 & 4 \end{vmatrix}$
3. $\begin{vmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{vmatrix}$
4. What positive value of x makes the following pair of determinants equal?

$\begin{vmatrix} 2x & 3 \\ 5 & x \end{vmatrix}$, $\begin{vmatrix} 16 & 3 \\ 5 & 2 \end{vmatrix}$

5. If A =$\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$ and B = $\begin{bmatrix} 0 & 4 \\ -1 & 7 \end{bmatrix}$, find the determinant of the matrix $3A^2 - 2B$
6. Evaluate : $\begin{vmatrix} 0 & 2 & 0 \\ 2 & 3 & 4 \\ 4 & 5 & 6 \end{vmatrix}$?
7. Evaluate $\begin{vmatrix} \cos\alpha\cos\beta & 2\cos\alpha\sin\beta & -\sin\alpha \\ -\sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha \end{vmatrix}$?
8. Find the integral value(s) of x if $\begin{vmatrix} x^2 & x & 1 \\ 0 & 2 & 1 \\ 3 & 1 & 4 \end{vmatrix}$ = 28
9. If A = $\begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}$then show that $|2A|$ = 4 $|A|$
10. Find the cofactors of the elements of the third row of the determinant and verify that $a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33} = 0$

$$\begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$$

11.Write the minors and cofactors of elements of determinants:

i. $\begin{vmatrix} 5 & 20 \\ 0 & -1 \end{vmatrix}$

ii. $\begin{vmatrix} 0 & 2 & 6 \\ 1 & 5 & 0 \\ 3 & 7 & 1 \end{vmatrix}$

iii. $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$

Applications of determinants :

Area of a triangle = $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

Exercise

1. Find the area of the triangle whose vertices are
(-8,-2) , (-4,-6),(-1,-5)
2. Using determinant show that the points are collinear.
(11,7), (5,5),(-1,3)

Properties of determinants:

1. If each entry of a row (or a column) of a determinant is zero then the value of the determinant is zero
2. If the rows are changed into columns and column into rows of a determinant then the value of the determinant remains unaltered.
3. If any two rows (or columns) of a determinant are interchange then the value of the determinant is the negative of the original determinant.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = - \begin{vmatrix} d & e & f \\ a & b & c \\ g & h & i \end{vmatrix}$$

4. If two rows (or columns) of a determinant are identical then the value of the determinant is zero.

5. If all the elements of a row(or column) of a determinant is multiplied with a non zero real number k, then the value o f the new determinant is k times the value of the original determinant.

6. If each entry of a row(or column) of a determinant is written as the sum of two or more terms , then the determinant can be written as sum of two or more determinants.

$$\begin{vmatrix} a+k & b & c \\ d+l & e & f \\ g+m & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} k & b & c \\ l & e & f \\ m & h & i \end{vmatrix}$$

7. If in any row or column of a determinant is multiplied with a constant and the resulting product is then added to any corresponding row or column in the determinant then the value of the determinant is equal to the original determinant.

$$\Delta \quad \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \quad , \quad \Delta_1 = \begin{vmatrix} a & b & c \\ d & e & f \\ g+ka & h+kb & i+kc \end{vmatrix}$$

Then $\Delta = \Delta_1$

8. If to each element of a line row or column of a determinant be added the equi-multiples of the corresponding elements of one or more parallel lines, then the determinant remains unaltered.ie.

$$\begin{vmatrix} a+lb+mc & b & c \\ d+le+mf & e & f \\ g+lh+mi & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

9. (Based on factor theorem) if the elements of a determinant that involves x are polynomials in x, and if the determinant is equal to 0 when a is substituted for x , then x – a is a factor of the determinant.
10. Product of two determinants.

Exercise

1. Without actually expanding the determinant but stating and using the theorems on determinants, show that

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 1 & 0 & -1 \end{vmatrix}$$

2. Without expanding the determinanant show that

i. $$\begin{vmatrix} 42 & 1 & 6 \\ 28 & 7 & 4 \\ 14 & 3 & 2 \end{vmatrix} = 0$$

ii. $$\begin{vmatrix} x+y & x & x \\ 5x+4y & 4x & 2x \\ 10x+8y & 8x & 3x \end{vmatrix} = x^3$$

Using the properties of the determinants , show that

i. $$\begin{vmatrix} 0 & p-q & p-r \\ q-p & 0 & q-r \\ r-p & r-q & 0 \end{vmatrix} = 0$$

ii. $$\begin{vmatrix} x+\lambda & x & x \\ x & x+\lambda & x \\ x & x & x+\lambda \end{vmatrix} = \lambda^2 (3x + \lambda)$$

Solve the following equations :

i. $$\begin{vmatrix} x-1 & 1 & 1 \\ 1 & x-1 & 1 \\ 1 & 1 & x-1 \end{vmatrix} = 0$$

ii. $$\begin{vmatrix} 1 & 4 & 20 \\ 1 & -2 & 5 \\ 1 & 2x & 5x^2 \end{vmatrix} = 0$$

Using properties of determinant ,prove that

1.

$$\begin{vmatrix} x & a & a \\ a & x & a \\ a & a & x \end{vmatrix} = (x + 2a)(x - a)^2$$

2.

$$\begin{vmatrix} a & a^2 & b+c \\ b & b^2 & a+c \\ c & c^2 & a+b \end{vmatrix} = (b-c)(c-a)(a-b)(a + b + c)$$

3.

$$\begin{vmatrix} x+a & b & c \\ a & x+b & c \\ a & b & x+c \end{vmatrix} = x^2(x + a + b + c)$$